A converse Lyapunov theorem for almost sure stabilizability *

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Abstract

We prove a converse Lyapunov theorem for almost sure stabilizability and almost sure asymptotic stabilizability of controlled diffusions: given a stochastic system a.s. stochastic open loop stabilizable at the origin, we construct a lower semicontinuous positive definite function whose level sets form a local basis of viable neighborhoods of the equilibrium. This result provides, with the direct Lyapunov theorems proved in a companion paper, a complete Lyapunov-like characterization of the a.s. stabilizability.

Key words. Degenerate diffusion, almost sure stability, stabilizability, asymptotic stability, stochastic control, control Lyapunov function, viscosity solutions, Hamilton-Jacobi-Bellman inequalities, viability.

1 Introduction

In this paper we provide a Lyapunov characterization of almost sure stochastic open loop stability at an equilibrium of controlled diffusion processes in $\mathbb{R}^N$

$$(CSDE)\begin{cases} dX_t = f(X_t,\alpha_t)dt + \sigma(X_t,\alpha_t)dB_t, \quad \alpha_t \in A, \quad t>0, \\ X_0 = x, \end{cases}$$

This notion has been introduced in a companion paper [6] by Bardi and the author (see also [5]): we say that $(CSDE)$ is *a.s. (open loop) stabilizable* if for any $\eta > 0$ there exists $\delta > 0$ such that, for any x with $|x| \le \delta$, there exists α such that the corresponding process satisfies $|X_t| \le \eta$ for all $t \ge 0$ almost surely. If, in addition, the trajectory is asymptotically approaching a.s. the equilibrium, we say the system is *a.s. (open loop) asymptotically stabilizable*. The definitions imply in particular that these properties are never verified by nondegenerate processes. This stochastic stability describes a behaviour very similar to a stable deterministic system and is stronger than pathwise stability and stability in probability (see [17, 19, 21]). We characterize it by means of appropriate control *Lyapunov functions*. These functions have been introduced in [5] and are lower semicontinuous (LSC), continuous at the equilibrium, positive definite, proper. Moreover they satisfy the following *infinitesimal decrease condition*:

$$\sup_{\sigma(x,\alpha)^T DV(x)=0} \left\{ -DV(x)\cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} \ge l(x), \tag{1}$$

where $a := \sigma\sigma^T/2$, $l \equiv 0$ for Lyapunov functions and positive definite for strict Lyapunov functions. This is not a standard Hamilton-Jacobi-Bellman inequality, because the constraint on the controls

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depends on V: we are allowing diffusion only in the directions tangential to the sublevel sets of V. If we eliminate this constraint, the differential inequality which is left is the infinitesimal decrease condition on Lyapunov functions for the stability in probability (see [12],[11]). We prove that V satisfies (1) if and only if it satisfies the following *monotonicity condition:*

$$\forall x \ \exists \alpha \ : \quad \sup_{t \geq 0} \operatorname*{ess\,sup}_{\omega \in \Omega} \left(V(X_t) + \int_0^t l(X_s) ds \right) \leq V(x) \tag{2}$$

where the essential supremum is intended with respect to the probability measure $\mathbf{P}_x$. This means in particular that the process $V(X_t)$ is a positive *supermaxingale* according to the definition given in [10]: this is the natural counterpart of the requirement on the process $V(X_t)$ to be a positive supermartingale in the context of stability in probability. The monotonicity condition says that the sublevel sets $K_\mu := \{x \ \ V(x) \leq \mu\}$ are *viable (or weakly invariant)* with respect to (CSDE) in the sense that $\forall \ x \in K_\mu \ \exists \alpha$ such that $X_t \in K_\mu$ forever almost surely. One of the main tool used in this paper is the geometric Nagumo-type characterization of viability proved recently by Bardi and Jensen in [9] (see also [8], [2] and the references therein for earlier related results).

In [6], Bardi and the author show that the existence of a Lyapunov function (respectively, of a strict Lyapunov function) implies the a.s. stabilizability (respectively, the a.s. asymptotic stabilizability) of the system to the equilibrium. As a simple example of application of this theory, we consider a radial function $V(x) = v(|x|)$, for some real smooth function v with $v'(r) > 0$ for $r > 0$. The system (CSDE) admits V as Lyapunov function if

$$\forall x \ \exists \alpha : \ \sigma(x, \alpha) \cdot x = 0 \quad f(x, \alpha) \cdot x + \mathit{trace} \ a(x, \alpha) \leq 0.$$

Therefore the following conditions are sufficient for the a.s. stabilizability: the radial component of the diffusion is null and its rotational component, which still plays a destabilizing role since $\mathit{trace}\, a(x, \alpha) \geq 0$, must be compensated by a negative radial component of f.

In this paper we prove that the existence of a Lyapunov function is also a necessary condition for a.s. stabilizability (and also for a.s. Lagrange stabilizability in the global case). A Lyapunov function for the system can be defined as

$$V(x) := \inf\{r \mid \exists \overline{\alpha} \text{ admissible control such that } |\overline{X}_t| \leq r \text{ almost surely } \forall t \geq 0\},$$

or equivalently as

$$V(x) := \inf_{\alpha \in \mathcal{A}_x} \sup_{t \geq 0} \operatorname*{ess\,sup}_{\omega \in \Omega} |X_t^\alpha|.$$

We prove that this function is LSC, continuous at the origin, positive definite, proper, and satisfies the infinitesimal decrease condition (1) with $l \equiv 0$. For an a.s. asymptotic stabilizable systems in a bounded set $\mathcal{O}$, we build a positive definite Lipschitz continuous function l, related to the rate of decrease of the stable trajectories to the equilibrium, by the formula

$$V(x) := \inf_{\alpha \in \mathcal{A}_x} \operatorname*{ess\,sup}_{\omega \in \Omega} \int_0^{+\infty} l(X_t^\alpha) dt.$$

We show that V is finite, LSC, continuous at the origin, positive definite, proper, and satisfies the infinitesimal decrease condition (1).

In both cases the Lyapunov functions are *worst-case value function* of an appropriate stochastic optimal control problem: we minimize the worst possible cost over all possible paths. This is quite natural since these functions characterize a very strong stability notion. The link between worst-case value functions and viscosity solutions to geometric second order partial differential equations has been recently treated by Soner and Touzi in [25] (see also [10]). They considered stochastic target problems where the controller tries to steer almost surely a controlled process into a given target by judicial choices of controls. The interest in this kind of stochastic control problems in the *almost sure* setting comes from the relationship with mean curvature type geometric flows and from the applications to the super-replication problems in financial mathematics. Moreover the

a.s. stability of control systems affected by disturbances modelled as M-dimensional white noise is related to the so-called *worst-case stability (or robust stability)* of deterministic control systems affected by disturbances modelled as (deterministic) L^∞ functions with values in $I\!R^M$ (see [16]). The Lyapunov characterization of these two stability properties seems to be an useful tool to give a precise proof of this relationship (see [12]), while a direct proof based on the estimates among the trajectories of the two systems should be rather hard. An approach of this type has been used recently by Da Prato and Frankowska in [14] to prove the equivalence between the invariance with respect to a controlled stochastic system and the invariance with respect to a deterministic system with two (non competitive) controls.

We conclude with some additional references on converse Lyapunov theorems. For controlled deterministic systems, there are theorems characterizing the stochastic open loop stabilizability by means of LSC appropriate Lyapunov functions (see [3]). Soravia in [28] showed that the stability at an equilibrium is equivalent to the continuity at such point of the value function $V(x) = \inf_{\alpha \in \mathcal{A}_x} \sup_{t \geq 0} U(X_t)$ where the level sets of U form a local basis of neighborhoods of the equilibrium. For asymptotically controllable systems, Sontag and Sussmann ([26], [27]) provided a characterization of asymptotic controllability by means of continuous Lyapunov functions such as $V(x) = \inf_\alpha \int_0^{+\infty} l(X_t)dt$ where l is an appropriate positive definite function. Recently Rifford ([23]) proved a converse Lyapunov theorem in the framework of Lipschitz continuous functions which are semiconcave outside the equilibrium. In the stochastic setting, Has'minskii ([17], [19]) obtained a converse theorem for stability in probability of uncontrolled diffusion processes, strictly nondegenerate outside the equilibrium, by means of $\mathcal{C}^2$ Lyapunov functions, using the Maximum Principle and the properties of solutions of uniformly elliptic equation. Kushner proved in [20] a characterization of asymptotic uniform stochastic stability by means of continuous Lyapunov functions (here, however, the infinitesimal decrease condition is given in terms of the weak generator of the process). In the forthcoming paper [11] (see also [12]) the author extends the direct Lyapunov method by Has'minskii and Kushner to the study of stochastic open loop stabilizability in probability in terms of merely semicontinuous Lyapunov functions which satisfy in the viscosity sense an appropriate infinitesimal decrease condition and provides also in this setting converse Lyapunov theorems.

The paper is organized as follows. In Section 2 we give the definition of stochastic open loop a.s. stabilizability, in Section 3 we introduce the appropriate concept of Lyapunov function for the study of such stability. Section 4 is devoted to the viability properties of sublevel sets of Lyapunov functions. Section 5 contains the main results: the converse Lyapunov theorems. Finally in Section 6 we give the extension to general equilibrium sets.

2 Almost sure Lyapunov stabilizability

We consider a controlled Ito stochastic differential equation:

$$(CSDE) \begin{cases} dX_t = f(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t, & t > 0, \\ X_0 = x. \end{cases}$$

where B_t is an M-dimensional Brownian motion. Throughout the paper we assume that f, σ are continuous functions defined in $I\!R^N \times A$, where A is a compact metric space, which take values, respectively, in $I\!R^N$ and in the space of $N \times M$ matrices, and satisfying for all $x, y \in I\!R^N$ and all $\alpha \in A$

$$|f(x, \alpha) - f(y, \alpha)| + \|\sigma(x, \alpha) - \sigma(y, \alpha)\| \leq C|x - y|, \tag{3}$$

We define $a(x, \alpha) := \frac{1}{2}\sigma(x, \alpha)\sigma(x, \alpha)^T$ and assume

$$\{(a(x, \alpha), f(x, \alpha)) \; : \; \alpha \in A\} \quad \text{is convex for all } x \in I\!R^N. \tag{4}$$

The class of admissible controls is the class of *strict controls*, as defined in [18, Definition 2.2]: they are A valued, progressively measurable processes α_t such that there exists a solution X_t^α to $(CSDE)$. $\mathcal{A}_x$ denotes the class of admissible control for a given initial datum x, with $\alpha.$ its generic

element (although it is not a standard function $I\!R \to A$), and with $X.$ the corresponding solution of $(CSDE)$. We recall also a theorem on the existence of optimal control for stochastic control problems.

Theorem 1 (Theorem 4.7 and Corollary 4.8 [18]). *Under the convexity assumption (4), for every initial data $x \in I\!R^N$ there exists an admissible control realizing the minimum in the control problem $\inf_\alpha \mathbf{E} J(x, \alpha)$ where the cost functional $J(x, \alpha)$ satisfies standard regularity assumptions.*

We state now the definition of almost sure stochastic open loop stabilizability, which has been introduced and studied in the paper [6] (see also [5] for the uncontrolled case). We introduce the classes of comparison functions.

Definition 2 (comparison functions). $\mathcal{K}$ denotes the class of real continuous functions γ strictly increasing and such that $\gamma(0) = 0$; $\mathcal{K}_\infty$ contains the functions $\gamma \in \mathcal{K}$ such that $\lim_{r\to+\infty} \gamma(r) = +\infty$. Finally $\mathcal{KL}$ denotes the class of continuous functions $\beta : I\!R \times I\!R \to I\!R$ which are strictly increasing in the first variable, strictly decreasing in the second variable, which satisfy $\beta(0, t) = 0$ for $t \geq 0$, $\lim_{t\to+\infty} \beta(r, t) = 0$ for $r \geq 0$.

Definition 3 (a.s. stabilizability). *The system* $(CSDE)$ *is* almost surely (open-loop Lyapunov) stabilizable *at the origin if there exists $\gamma \in \mathcal{K}$ and $\delta_o > 0$ such that for any starting point x with $|x| \leq \delta_o$*

$$\exists\, \overline{\alpha}. \in \mathcal{A}_x : \quad |\overline{X}_t| \leq \gamma(|x|) \quad \forall t \geq 0 \ \ a.s. \tag{5}$$

If γ can be chosen in $\mathcal{K}_\infty$ and the estimate (5) holds in the whole space $I\!R^N$, the system is also almost surely (open-loop) Lagrange stabilizable*, or it has the property of* uniform boundedness of trajectories*.*

Remark 4. We could define the a.s. stabilizability equivalently as follows:
the system is a.s. (open loop) stabilizable at the origin if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $|x| \leq \delta$ there exists an admissible control function $\overline{\alpha}. \in \mathcal{A}_x$ whose corresponding trajectory $\overline{X}.$ verifies $|\overline{X}_t| \leq \varepsilon$ for all $t \geq 0$ almost surely.

Remark 5. A necessary condition for the a.s. stabilizability at the origin is that the origin is a *controlled equilibrium* of $(CSDE)$, i.e.,

$$\exists\, \overline{\alpha} \in A \, : \, f(0, \overline{\alpha}) = 0, \ \sigma(0, \overline{\alpha}) = 0. \tag{6}$$

Definition 6 (a.s. asymptotic stabilizability). *The system* $(CSDE)$ *is* almost surely (open loop) locally asymptotically stabilizable *(or* a. s. locally asymptotically controllable*) at the origin if there is $\beta \in \mathcal{KL}$ and $R > 0$ such that for any starting point x with $|x| \leq R$ there exists $\overline{\alpha}. \in \mathcal{A}_x$*

$$|\overline{X}_t| \leq \beta(|x|, t) \quad \forall t \geq 0 \ \ a.s. \tag{7}$$

3 Lyapunov functions for a.s. stabilizability

In this section we introduce the appropriate concept of Lyapunov function for the study of the almost sure stochastic stability.
We recall the definition of the second order semijet (see [13]) of a LSC function V at x $\mathcal{J}^{2,-}V(x) := \{(p, Y) \in I\!R^N \times S(N)$ such that for $y \to x$, $V(y) \geq V(x) + p\cdot(y-x) + \frac{1}{2}(y-x)\cdot Y(y-x) + o(|y-x|^2)\}$.

Definition 7 (control Lyapunov function). *Let $\mathcal{O} \subseteq I\!R^N$ be an open set containing the origin. A function $V : \mathcal{O} \to [0, +\infty)$ is a* control Lyapunov function *for the a.s. stabilizability of* $(CSDE)$ *if*
(i) V is lower semicontinuous and continuous at 0;
(ii) V is positive definite*, i.e., $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$;*
(iii) V is proper*, i.e., the sublevel sets $\{x|V(x) \leq \mu\}$ are bounded $\forall \mu \in [0, \infty)$;*

(iv) V is a viscosity supersolution in $\mathcal{O} \setminus \{0\}$ of the equation:

$$\sup_{\sigma(x,\alpha)\cdot DV(x)=0} \left\{-DV(x) \cdot f(x,\alpha) - trace[a(x,\alpha)D^2V(x)]\right\} \geq 0, \tag{8}$$

in the following sense: for all $x \in \mathcal{O} \setminus \{0\}$ and $(p,Y) \in \mathcal{J}^{2,-}V(x)$ there exists $\overline{\alpha} \in A$:

$$\sigma(x,\overline{\alpha})^T p = 0 \qquad and \qquad - p \cdot f(x,\overline{\alpha}) - trace\,[a(x,\overline{\alpha})Y] \geq 0.$$

If there exists a positive definite, Lipschitz continuous $l : \mathcal{O} \to I\!R$ such that

$$\sup_{\sigma(x,\alpha)\cdot DV(x)=0} \left\{-DV(x) \cdot f(x,\alpha) - trace[a(x,\alpha)D^2V(x)]\right\} \geq l(x) \tag{9}$$

then V is a strict control Lyapunov function *for the a.s. stabilizability of $(CSDE)$.*

The inequality (8) is not the standard Hamilton-Jacobi-Bellman inequality arising in stochastic optimal control. We have an implicit constraint on the controls, $\sigma(x,\alpha)\cdot DV(x) = 0$, i.e. depending on the generalized subgradients of the solution: we are allowing only controls which render the diffusion matrix tangential in some generalized sense to the sublevel sets of V. This implies that the diffusion has to degenerate in a large set, for some control.

Because of this constraint, though, the nonlinearity

$$F(x,p,X) = \sup\left\{-p \cdot f(x,\alpha) - trace[a(x,\alpha)X] \mid a \in A \ \sigma(x,\alpha) \cdot p = 0\right\}$$

is **geometric** in the sense that it satisfies the following rescaling property $F(x, \lambda p, \lambda X + \mu p \otimes p) = \lambda F(x,p,X)$ for every $\lambda > 0$ and $\mu \in I\!R$, where $p \otimes p$ is the $N \times N$ matrix whose (i,j) entry is $p_i p_j$. This permits to prove the following lemma on the change of unknown (for the proof see [6], [12]).

Lemma 8. *Assume that v is a LSC viscosity supersolution of equation (8) in an open set $\mathcal{O}$. Let ϕ be a twice continuously differentiable strictly increasing real map. Then $w = \phi \circ v$ is still a viscosity supersolution of equation (8) in $\mathcal{O}$.*

4 Viability properties of Lyapunov functions

We study now a viability property of the sublevel sets of viscosity supersolution of the nonstandard Hamilton-Jacobi-Bellman inequality (8). We recall the definition of almost sure viability (named also *controlled invariance* and *weak invariance*) of an arbitrary closed set for a controlled diffusion process.

Definition 9 (viable set). *A closed set $K \subset I\!R^N$ is* viable *or* controlled invariant *or* weakly invariant *for the stochastic system $(CSDE)$ if for all initial points $x \in K$ there exists an admissible control $\alpha_\cdot \in \mathcal{A}_x$ such that the corresponding trajectory $X_\cdot$ satisfies $X_t \in K$ for all $t > 0$ almost surely.*

This property was studied by Aubin and Da Prato [2] and, more recently, by Bardi and Jensen [9]. The main result of [9] is the equivalence between the viability of a closed set K and a Nagumo-type geometric condition. This geometric condition is given in terms of the *second order normal cone* to a closed set $K \subset I\!R^N$, first introduced in [8],

$$\begin{aligned} \mathcal{N}^2_K(x) \ &:= \ \{(p,Y) \in I\!R^N \times S(N) : \ \text{for } \ y \to x, \ \ y \in K, \\ &\qquad p \cdot (y-x) + \frac{1}{2}(y-x) \cdot Y(y-x) \geq o(|y-x|^2) \ \ \} \end{aligned}$$

where $S(N)$ is the set of symmetric $N \times N$ matrices. Note that, if ∂K is a smooth surface in a neighborhood of x, $p/|p|$ is the interior normal and Y is related to the second fundamental form of ∂K at x, see [8].

Theorem 10 (Viability theorem [9]). *Assume conditions (3) and (4). Then K is viable for $(CSDE)$ if and only if*

$$\forall x \in \partial K,\ \forall (p,Y) \in \mathcal{N}^2_K(x),\ \exists \alpha \in A\ :\ \ f(x,\alpha)\cdot p + trace\,[a(x,\alpha)Y] \geq 0.$$

Moreover, for the same α we have that $\sigma(x,\alpha)\cdot p = 0$.

Using this result we obtain the following characterization, which is a variant of a result contained in [6] (see also [5]).

Lemma 11. *Assume conditions (3) and (4). Consider an open set $\mathcal{O} \subseteq I\!R^N$ and a LSC function $V : \mathcal{O} \to I\!R$. If V is a viscosity supersolution of*

$$\sup_{\sigma(x,\alpha)\cdot DV=0} \left\{ -DV\cdot f(x,\alpha) - trace[a(x,\alpha)D^2V] \right\} \geq 0 \tag{10}$$

in $\mathcal{O}$, then the sublevel sets $\{V(x) \leq \mu\}$ whose boundary is entirely contained in $\mathcal{O}$ are viable with respect to (CSDE). Viceversa, suppose $\overline{\mu}$ is the maximal value for which the sublevel set $\{V(x) \leq \mu\}$ has boundary entirely contained in $\mathcal{O}$. Then the function $\overline{V}(x) := V(x) \wedge \overline{\mu}$ is a viscosity supersolution of (10) in $\mathcal{O}$.

Observe that if $\mathcal{O} = I\!R^N$, every sublevel set of V is closed (in particular we can take $\overline{\mu} = +\infty$).

Proof. For every $\mu \leq \overline{\mu}$, $K_\mu := \{x \mid V(x) \leq \mu\}$. We define now the LSC function $V_\mu(x) := \mu$ for $x \in K_\mu$ and $+\infty$ elsewhere. From the definitions, it is easy to check that $\mathcal{J}^{2,-}V_\mu(x) = -\mathcal{N}^2_K(x), \quad \forall\, x \in \partial K_\mu$, so, by the Viability Theorem 10, V_μ is a viscosity supersolution of (10) if and only if K_μ is viable.

We assume that V is a viscosity supersolution of (10). Now for $\lambda > 0$ fixed, we define the nondecreasing continuous real function

$$\psi_\lambda(t) = \begin{cases} \mu, & t \leq \mu, \\ \lambda^2(t-\mu) + \mu, & \mu \leq t \leq \mu + \frac{1}{\lambda}, \\ \lambda + \mu & t \geq \mu + \frac{1}{\lambda}. \end{cases}$$

The function $\psi_\lambda \circ V$ is a viscosity supersolution of equation (10) in $\mathcal{O}$ for every λ. To prove this fact, we choose a sequence ψ_n of strictly increasing, smooth real maps that converge uniformly on compact sets to ψ_λ. Then, for every n, the map $\psi_n \circ V$ is a viscosity supersolution of equation (10) in $\mathcal{O}$ by Lemma 8. This permits to conclude, by the stability of viscosity supersolutions with respect to uniform convergence. Next we observe that the net $\psi_\lambda \circ V$ is increasing and converges as $\lambda \to +\infty$ to V_μ. Viscosity supersolutions are stable with respect to the pointwise increasing convergence (see [4]). Therefore the indicator function V_μ of K_μ is a viscosity supersolution of equation (10) and then K_μ is viable for $\mu \leq \overline{\mu}$.

Conversely, we assume now that K_μ is viable for every $\mu \leq \overline{\mu}$. Moreover, we observe that

$$\overline{V}(x) = \inf_{\mu \leq \overline{\mu}} V_\mu(x) \wedge \overline{\mu} = \inf\{\mu \leq \overline{\mu} \mid V(x) \leq \mu\} \wedge \overline{\mu}.$$

So, by the stability properties of viscosity supersolutions, if V_μ solves (10) for every $\mu \leq \overline{\mu}$, then $\overline{V}$ solves (10) too. $\square$

This lemma provides the main tool to prove the direct Lyapunov theorems (see [6], [12]).

Theorem 12 (Direct Lyapunov theorem). *Assume (3), (4). If the system admits a Lyapunov function in an open set $\mathcal{O}$ containing the equilibrium then*
(i) the system $(CSDE)$ is almost surely stabilizable at the origin;
(ii) if, in addition, the domain $\mathcal{O}$ can be chosen as $I\!R^N$, the system is also a.s. Lagrange stabilizable and for all $x \in I\!R^N$ there exists $\overline{\alpha}_\cdot \in \mathcal{A}_x$ such that the corresponding trajectory $\overline{X}_\cdot$ satisfies

$$|\overline{X}_t| \leq \gamma_1^{-1}(\gamma_2(|x|)) \quad \forall\, t \geq 0 \quad a.s. \tag{11}$$

with $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$.
Assume moreover that V is a strict Lyapunov function then
(i) the system $(CSDE)$ is a.s. locally asymptotically stabilizable at the origin;
(ii) if, in addition, the domain $\mathcal{O}$ of V is all $I\!R^N$, the system is a.s. globally asymptotically stabilizable.

5 Converse Lyapunov theorems for a.s. stabilizability

In this section we prove the main results in the article: we assume that the system (CSDE) satisfies an a.s. stabilizability property and construct an appropriate Lyapunov function.

Theorem 13 (a.s. stabilizability). *Assume (3), (4). Then*
(i) if the system $(CSDE)$ is almost surely stabilizable at the origin in the ball B_K, the function

$$V(x) = \left[\inf_{\alpha \in \mathcal{A}_x} \sup_{t \geq 0} ess \sup_{\omega \in \Omega} |X_t^\alpha| \right] \wedge K$$

is a Lyapunov function for the system in B_K;
(ii) if the system is also a.s. Lagrange stabilizable then the function

$$V(x) = \inf_{\alpha \in \mathcal{A}_x} \sup_{t \geq 0} ess \sup_{\omega \in \Omega} |X_t^\alpha|$$

is a global Lyapunov function for the system.

Proof. We start proving (i). We start constructing the larger viable set containing the origin in the ball $\overline{B_K}$. We construct a nonnegative, uniformly continuous, radial function c_K such that $c_K(|s|) = 0$ if $|s| \leq K$ and $0 < c_K(|s|) \leq |s|$ for $|s| > K$ and for $\lambda < 0$ we consider the value function

$$W_K(x) = \inf_{\alpha} \mathbf{E}_x \int_0^{+\infty} c_K(|X_s^\alpha|) e^{-\lambda s} ds.$$

It is well known ([22] and [15]) that W_K is a continuous viscosity supersolution of the Hamilton-Jacobi-Bellman equation in $I\!R^N$

$$\max_{\alpha \in A} \left\{ -DW(x) \cdot f(x, \alpha) - trace\left[a(x, \alpha) D^2 W(x) \right] \right\} + \lambda W(x) \geq c_K(|x|)$$

The function W_K is nonnegative in $I\!R^N$ and $W_K(0) = 0$ since the origin is a controlled equilibrium as remarked in (6). We consider the propagation set of the minimum value 0:

$$Prop(0, W_K) = \{x \in I\!R^N \mid W_K(x) = 0\} = \{x \mid \exists \overline{\alpha} \ : \ \overline{X}_t \in \overline{B_K} \ a.s. \forall t\} \subseteq \overline{B_K}.$$

This set is clearly closed and it can be proved that it is also viable. This follows immediately from Theorem 4.6 in Bardi, Da Lio [7] or can be checked directly.

The candidate Lyapunov function is defined in the set $\overline{\mathcal{O}} = \overline{B_K}$ as

$$V(x) = \begin{cases} \inf_{\alpha \in \mathcal{A}_x} \sup_{t \geq 0} \operatorname{ess\,sup}_{\omega \in \Omega} |X_t^\alpha| & x \in Prop(0, W_K) \\ K & x \in \overline{B_K} \setminus Prop(0, W_K). \end{cases}$$

First of all we observe that the function V is well defined: by definition of a.s. stabilizability, there exists a function $\gamma \in \mathcal{K}$ such that $V(x) \leq \gamma(|x|)$. From this we get also that V is continuous at the origin. We observe also that $Prop(0, W_K) \supseteq B(0, \gamma^{-1}(K))$. Moreover V is positive definite. Indeed if $V(x) = 0$ then for every $\varepsilon > 0$ there exists α_ε such that the corresponding trajectory satisfies $|X_t| \leq \varepsilon$ for all $t \geq 0$ almost surely: so $\inf_{\alpha_. \in \mathcal{A}_x} \mathbf{E}_0 \int_0^{+\infty} |X_t| e^{-\lambda t} dt = 0$ for any fixed $\lambda > 0$. By Theorem 1, the inf is attained, and the minimizing control produces a trajectory satisfying a.s. $|X_t| = 0$ for all $t \geq 0$.

To prove the semicontinuity and the differential inequality, we provide another characterization of V. We prove that it coincides with

$$w(x) := \inf\{r \mid \exists \overline{\alpha} \in \mathcal{A}_x \ |\overline{X}_t| \leq r \quad \forall t \geq 0 \quad a.s.\} \wedge K. \tag{12}$$

First of all with the usual argument based on Theorem 1, we get that this infimum is actually a minimum. Then we fix $K \geq k > w(x)$: by definition there exists $\overline{\alpha} \in \mathcal{A}_x$ such that $|\overline{X}_t| \leq k - \varepsilon$ for all $t \geq 0$ a.s. This means that $\sup_{t\geq 0} \operatorname{ess\,sup}_{\omega\in\Omega} |\overline{X}_t| < k$ from which we deduce $w(x) \geq V(x)$. The converse is similar.

Therefore, for every $0 \leq \mu < K$, the sublevel set $\{x \in \mathcal{O} \mid V(x) \leq \mu\}$ coincides with $\{x \in I\!R^N \mid \exists \overline{\alpha} \in \mathcal{A}_x \ |\overline{X}_t| \leq \mu \ \ \forall t \geq 0 \ \ a.s.\}$; in particular this gives that the function V is proper. As at the beginning of the proof, we can characterize these sets as the propagation sets of the minima of viscosity supersolutions of suitable Hamilton-Jacobi-Bellman equations. Then, by the results in [7], these sets are closed in $I\!R^N$ and viable with respect to (CSDE). So the function V is LSC. Moreover, by Lemma 11, it satisfies the infinitesimal decrease condition (8).

To prove (ii), we observe that for every $K > 0$ we can repeat the previous construction, since the estimate (5) holds in the whole space with $\gamma \in \mathcal{K}_\infty$. So we get an increasing sequence of Lyapunov functions V_K: for every $K > 0$ we construct as before the function V_K in the ball B_K and extend it to the whole space in the obvious way. Hence the global Lyapunov function for the system is

$$V(x) = \lim_{K\to+\infty} V_K(x) = \sup_{K>0} V_K(x) = \inf_{\alpha\in\mathcal{A}_x} \sup_{t\geq 0} \operatorname*{ess\,sup}_{\omega\in\Omega} |X_t^\alpha|.$$

It is immediate to check that it satisfies the condition (i),(ii),(iii) in the Definition 7. Moreover the proof of the fact that V satisfies the differential condition (iv) relies on standard stability properties of viscosity supersolutions with respect to the pointwise increasing convergence ([4]). ☐

Now we prove the converse Lyapunov theorem for asymptotic stability in a bounded set.

Theorem 14 (a.s. asymptotic stabilizability). *Assume (3), (4). If the system (CSDE) is a.s. asymptotically stabilizable at the origin in the ball B_K, then there exist an open set $\mathcal{O}$ containing the origin and a Lipschitz continuous positive definite function $l : \mathcal{O} \to I\!R$ such that*

$$V(x) = \inf_{\alpha\in\mathcal{A}_x} \operatorname*{ess\,sup}_{\omega\in\Omega} \int_0^{+\infty} l(X_t^\alpha)dt$$

is a Lyapunov function for the system in $\mathcal{O}$.

Proof. The estimate (7) permits to construct a positive definite Lipschitz continuous function $l : I\!R^N \to I\!R$ as follows. $t_x(U)$ denotes the random time spent in the set U by the trajectory $\overline{X}$. Using the properties of the function β we get that, for every $r \leq K$ and $x \in B_K$, the time $t_x(I\!R^N \setminus B_r)(\omega)$ spent by the trajectory $\overline{X}$ outside B_r is almost surely bounded and moreover it satisfies

$$\sup_{x\in B_K} \operatorname*{ess\,sup}_{\omega\in\Omega} t_x(I\!R^N \setminus B_r)(\omega) < +\infty.$$

We consider now a decreasing sequence of positive numbers r_i such that $r_0 < K$ and $\lim_{i\to+\infty} r_i = 0$. We define

$$T_i = \sup_{x\in B_K} \operatorname*{ess\,sup}_{\omega\in\Omega} t_x(I\!R^N \setminus B_{r_i})(\omega).$$

The sequence of positive numbers T_i is increasing as $i \to +\infty$: we can choose a decreasing sequence of positive numbers l_i such that $\sum_{i=0}^{+\infty} l_i T_i = M < +\infty$. The function $l : I\!R^N \to I\!R$ is therefore defined as a radial Lipschitz continuous, positive definite, nondecreasing function which satisfies $l(0) = 0$, $l(|x|) = l_{i+1}$ for $|x| = r_i$ and $l(|x|) = l_1$ for every $|x| \geq r_0$.

The candidate Lyapunov function is

$$V(x) = \inf_{\alpha\in\mathcal{A}_x} \operatorname*{ess\,sup}_{\omega\in\Omega} \int_0^{+\infty} l(|X_t^\alpha|)dt \qquad \text{for } x \in B_K.$$

The rest of the proof will be devoted to show that this function satisfies the properties of Definition 7 and then is a strict Lyapunov function. First of all V is well defined:

$$V(x) = \inf_{\alpha \in \mathcal{A}_x} \operatorname*{ess\,sup}_{\omega \in \Omega} \int_0^{+\infty} l(|X_t^\alpha|)dt \leq \operatorname*{ess\,sup}_{\omega \in \Omega} \int_0^{+\infty} l(|\overline{X}_t|)dt \leq \sum_{i=0}^{+\infty} l_i T_i = M.$$

By definition $V(x) \geq 0$ for every x and $V(0) = 0$. We assume now that for some $x \neq 0$ $V(x) = 0$: this means that for every $\varepsilon > 0$ there exists $\alpha_\varepsilon \in \mathcal{A}_x$ such that $\int_0^{+\infty} l(|X_t^{\alpha_\varepsilon}|)dt \leq \varepsilon$ almost surely. Then $\inf_\alpha \mathbf{E}_x \int_0^{+\infty} l(|X_t^\alpha|)dt = 0$: from this, by the usual argument based on Theorem 1, we deduce that $x = 0$. We show now that the function V is continuous at the origin. Recalling the definition of $t_x(I\!R^N \setminus B_r)$ and the a.s. asymptotic stabilizability, we get that $t_x(I\!R^N \setminus B_r) = 0$ almost surely for initial data x such that $\beta(|x|, 0) \leq r$. Therefore, by the continuity at $r = 0$ of the function $\beta(r, 0)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for $|x| \leq \delta$, $\beta(|x|, 0) \leq \varepsilon$ and then $t_x(I\!R^N \setminus B_k) = 0$ almost surely for every $k \geq \varepsilon$. So we get $V(x) \leq \sum_{i=i(\varepsilon)}^{+\infty} l_i T_i$ where $i(\varepsilon)$ is the minimum index for which $r_{i(\varepsilon)} \leq \varepsilon$. Since the sum $\sum_i l_i T_i$ converges and $r_i \to 0$ as $i \to +\infty$, for every $\theta > 0$, we can choose $\varepsilon > 0$ such that $\sum_{i=i(\varepsilon)}^{+\infty} l_i T_i \leq \theta$: this gives the continuity at the equilibrium. To conclude the proof we have to provide another equivalent definition of V. We consider the new system in $I\!R^{N+1}$

$$(CSDE2) \begin{cases} d(X_t, Y_t) = \overline{f}(X_t, Y_t, \alpha_t)dt + \overline{\sigma}(X_t, Y_t, \alpha_t)d(B_t, 0), & t > 0, \\ (X_0, Y_0) = (x, y). \end{cases}$$

where $\overline{f}(x, y, \alpha) = (f(x, \alpha), l(x))$ and $\overline{\sigma}(x, y, \alpha) = (\sigma(x, \alpha), 0)$. It satisfies conditions (3) and (4) and has $(0, 0)$ as a controlled equilibrium. We introduce now the following function

$$W(x, y) = \inf\{r \mid \exists \alpha \in \mathcal{A}_x \ |Y_t^\alpha| \leq r \ \forall t \geq 0 \text{ a.s.}\} =$$

$$= \inf\{r \mid \exists \alpha \in \mathcal{A}_x \ \left| y + \int_0^{+\infty} l(|X_t^\alpha|)dt \right| \leq r \text{ a.s.}\}.$$

Using Theorem 1, it can be proved easily that this infimum is actually a minimum. So the sublevel sets of W are

$$\{(x, y) \mid W(x, y) \leq \mu\} = \left\{(x, y) \mid \exists \alpha \in \mathcal{A}_x \ \left| y + \int_0^{+\infty} l(|X_t^\alpha|)dt \right| \leq \mu \text{ a.s.} \right\}.$$

Repeating the argument in the proof of Theorem 13, we get that these sets are closed and viable with respect to (CSDE2). So the function $W(x, y)$ is LSC and satisfies, by Lemma 11, in viscosity sense

$$\sup_{\sigma(x,\alpha) \cdot D_x W = 0} \left\{ -D_x W \cdot f(x, \alpha) - trace[a(x, \alpha) D^2_{xx} W] \right\} - l(|x|) D_y W \geq 0. \tag{13}$$

Now we show that the candidate Lyapunov function V coincides with the function W on the set $B_K \times \{y = 0\}$. Assume that $V(x) \leq r$. For every $\varepsilon > 0$, there exists $\alpha_\varepsilon \in \mathcal{A}_x$ such that almost surely $\int_0^{+\infty} l(|X_t^{\alpha_\varepsilon}|)dt \leq r + \varepsilon$. Therefore $W(x, 0) \leq r + \varepsilon$ and so we conclude by the arbitrariness of ε. The proof of the opposite inequality $V(x) \leq W(x, 0)$ is similar. From this characterization of the function V we deduce immediately that it is LSC and bounded in B_K. It remains to check the differential condition (9). We fix $x \neq 0$ in B_K and consider $(p, Y) \in \mathcal{J}^{2,-}V(x)$: by definition, for every $x' \to x$, we get $W(x', 0) \geq W(x, 0) + p \cdot (x' - x) + 1/2(x' - x)Y(x' - x) + o(|x - x'|)$. Using the definition, it is immediate to check $W(x', 0) \leq W(x', y) - y$. This implies that if $(p, Y) \in \mathcal{J}^{2,-}V(x)$ then $(p, Y, 1) \in \mathcal{J}^{2,1,-}W(x, 0)$. So the differential condition (9) comes from (13). □

6 Extensions

We can extend the results to the characterization of stabilizability of a closed set $M \subseteq I\!R^N$. We denote with $d(x, M)$ the distance between $x \in I\!R^N$ and M.

Definition 15 (a.s. stabilizability at M). *The system* $(CSDE)$ *is* almost surely (open loop) stabilizable *at M if there exists $\gamma \in \mathcal{K}$ such that, for every x in a neighborhood of M, there is an admissible control function $\overline{\alpha}. \in \mathcal{A}_x$ whose trajectory $\overline{X}.$ verifies*

$$d(\overline{X}_t, M) \leq \gamma(d(x, M)) \quad \forall\, t \geq 0 \quad \textit{almost surely.}$$

Remark 16. From this definition, using Theorem 1, we can deduce that if M is a.s. stabilizable, then it is viable for $(CSDE)$.

We adapt the definition of control Lyapunov function to the case the equilibrium is a set M:

Definition 17 (control Lyapunov functions at M). *Let $\mathcal{O}$ be an open neighborhood of the closed set M. A function $V : \mathcal{O} \to [0, +\infty)$ is a control Lyapunov function at M for $(CSDE)$ if*
(i) V is lower semicontinuous;
(ii) there exist $\gamma_1, \gamma_2 \in \mathcal{K}$ such that $\gamma_2(d(x, M)) \leq V(x) \leq \gamma_1(d(x, M))$ for all $x \in \mathcal{O}$;
(iii) for all $x \in \mathcal{O} \setminus M$ and $(p, Y) \in \mathcal{J}^{2,-}V(x)$ there exists $\overline{\alpha} \in A$ such that condition (9) holds.

Theorem 18. *Assume (3), (4). Then the system $(CSDE)$ is almost surely stabilizable at M if and only if there exists a control Lyapunov function V at M.*

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